

# Exact solutions for characterization of electro-elastically graded materials

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## Abstract

The present paper deals with a class of functionally graded materials (FGM), called active FGM that has electro-elastically graded material phases. An active FGM system leads to minimization of stress concentration that arises due to mismatch in the electrical and elastic properties of the constituent phases. This work focuses on the characterization of the through thickness stresses of an active FGM subjected to electrical excitation. The structure is comprised of a substrate, an electro-elastically graded layer and an active layer. A formulation for exact solutions of the system based on Euler–Bernoulli theory is presented. Power-law variation of the composition of the two phases in the graded layer is considered. Performance of linearly gradient FGM for a range of stiffness and electrical property ratios of the active and substrate materials have been studied. It is observed that the electrical strain component and the compositional gradation significantly influence the stress characteristics of the active FGM.

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## 1. Introduction

Active materials are capable of responding to external stimuli by converting one form of energy (electrical, thermal, mechanical etc.) into another. A broad class of such materials viz. piezoceramics, piezopolymers, shape memory alloys etc. have been employed in controlling the response characteristics of structural systems that span over a wide range of length scales. Typically, these materials are bonded to or embedded into a host (passive) structure and their performance relies on

the efficacy of the bond between the two materials. However, when the active layer is excited using a particular energy form (e.g. electrical energy in case of piezoceramics), high stress concentration develops at the active–passive interface. The interface between the layers of these dissimilar materials is a potential weak zone where damage may occur. In such situations it is highly desirable to minimize the stress concentrations while simultaneously improving the response of the material to external stimulus. A solution to this problem is the development of a material interface with continuous gradation of the active–passive material. A graded active material would not only minimize the stress concentration but also be efficient in the strain transfer to the substrate due to better bi-material bond.

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Functionally gradient materials (FGM) have received considerable attention in recent years. Primary focus in the FGM development has been on materials suited for minimizing the deleterious effects arising due to dissimilar thermal expansion coefficients of the constituent materials [1]. An important application of graded materials is in tribology [2] in order to improve the damage resistance of contact surfaces. The response of graded materials under transient loads has been studied in order to distinguish the stress wave propagation in graded materials from layered materials [3,4]. In [4] a semi-empirical numerical model has been developed to include strain-rate effect in the elastic and viscoplastic response. Another domain of FGM research has been microstructural characterization. Significant amount of work is being carried out on determining material parameters such as elastic and plastic properties and fracture toughness [5–7]. While most studies are carried out for interphases with linear, power-law or exponential variations, some work has been conducted on randomly interpenetrating phases [8,9].

Employing concepts similar to the thermo-mechanically graded interphases, a graded active material can be engineered to take care of high anisotropy between the electro-elastic properties of the constituent phases and enhance the response characteristics of the structure. However, with improved functionality of the material, its characterization and optimization pose new challenges. It is highly desirable to study the response characteristics of graded active materials. In macrostructural analysis, study of variation of responses such as deformations, stresses etc. with different realistic electro-elastic gradations is warranted.

Active FGM is a nascent area of research. However, design of multilayered actuator in [10] indicates the importance of functionally graded piezoelectric materials in practical applications. The reader may refer to [11,12] for the process development for a typical active FGM. Analytical studies are focussed mainly on either elastic or electrical gradation [13–17]. In [13] microdisplacement and microstructure characteristics of graded piezoelectric materials have been reported.

Few researchers have studied controlled responses of functionally graded plates with bonded actuators [14,15]. In these cases, only elastic gradation is considered. In [16] exact solutions are derived for electrically graded monomorphs. The results indicate strong influence of electrical gradation on the stress characteristics and electric field. Recently, some numerical investigations for FGM bimorphs have been carried out for linear and nonlinear gradation [17] to maximize displacement of the actuator while minimizing the maximum stress.

In this paper, exact solutions are presented for obtaining the response of active FGM to electrical excitation using Euler–Bernoulli theory of bending. The through thickness stress distribution developed due to electrical effects with different polarities at the top and the bottom of the active layer is investigated. In the available literature, the power-law type compositional gradation has been frequently employed. The performance of this type of grading is evaluated vis-à-vis a complete polynomial type grading. The piezoelectric active layers are considered. Several substrate–active material ratios of practical interest are considered.

## 2. Material system

The material comprises of a symmetric lay-up of a piezoelectric layer, a graded layer and a substrate (Fig. 1).

In the present work, two gradation functions ( $N(z)$ ) are considered. They are defined in terms of the volume fraction of the active material in the graded layer. In the first case, a power-law function is considered such that

$$N(z) = \begin{cases} 0 & @z = h_1 \text{ and } z = -h_1 \\ 1 & @z = h_2 \text{ and } z = -h_2 \end{cases} \quad (1)$$

The power-law function is given as (Fig. 2)

$$N(z) = \left( \frac{z - h_1}{h_2 - h_1} \right)^n \quad (2)$$

In the second case, a generalized cubic shape function is considered as follows:

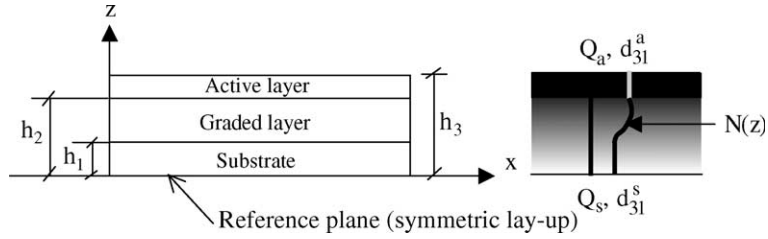


Fig. 1. Lay-up and property gradation of an electro-elastically graded material.

$$N(z) = a + b\left(\frac{z - h_1}{h_2 - h_1}\right) + c\left(\frac{z - h_1}{h_2 - h_1}\right)^2 + d\left(\frac{z - h_1}{h_2 - h_1}\right)^3 \quad (3)$$

In addition to the boundary conditions in Eq. (1), constraints are put on the gradient of the function as follows:

$$\frac{dN(z)}{dz} = \begin{cases} 0 & @z = h_1 \text{ and } z = -h_1 \\ 0 & @z = h_2 \text{ and } z = -h_2 \end{cases} \quad (4)$$

Substituting

$$\eta = \left(\frac{z - h_1}{h_2 - h_1}\right), \quad \text{where } 0 \leq \eta \leq 1$$

Therefore, Eq. (3) is written as

$$N(\eta) = a + b\eta + c\eta^2 + d\eta^3 \quad (5)$$

$$\frac{dN(\eta)}{d\eta} = b + 2c\eta + 3d\eta^2 \quad (6)$$

Using Eqs. (1) and (4)–(6), a set of simultaneous equations is written as

$$\begin{bmatrix} 1 & \eta & \eta^2 & \eta^3 \\ 0 & 1 & \eta & \eta^2 \\ 1 & \eta & \eta^2 & \eta^3 \\ 0 & 1 & \eta & \eta^2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad (7)$$

Solving Eq. (7) we get a cubic gradation with controlled interfacial gradients (CG2) (Fig. 3)

$$N(\eta) = (3\eta^2 - 2\eta^3) \quad (8)$$

The variation of material properties over FGM thickness is defined as

$$P(z) = (P_s + \Delta P N(z)) \quad (9)$$

where  $\Delta P = (P_a - P_s)$  and  $P_s$  and  $P_a$  are the material properties of the substrate and the active layer.

The constitutive relation that governs the electro-elastic behavior of a piezoelectric medium is given as

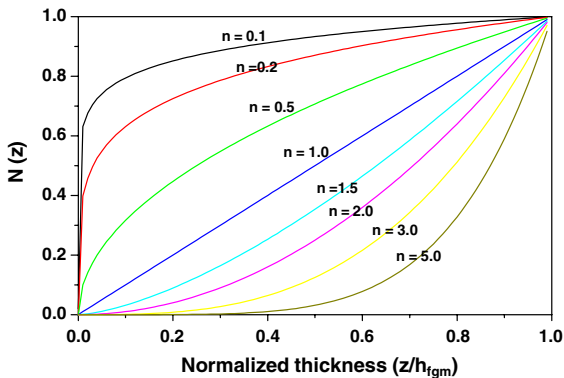


Fig. 2. Power-law variation of electro-elastic properties in the FG layer.

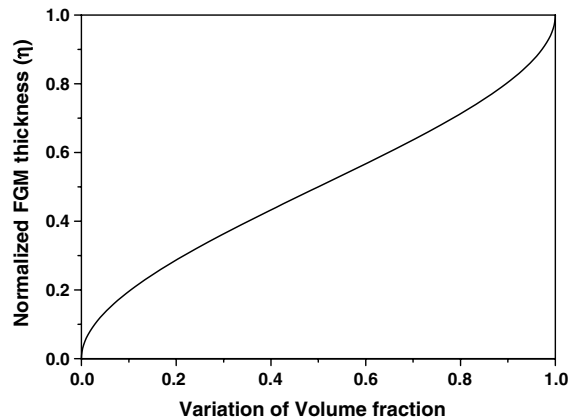


Fig. 3. CG2 variation of electro-elastic properties in the FG layer.

$$\sigma_{ij} = Q_{ijkl}(\varepsilon_{kl} - d_{klm}E_m) \quad (10)$$

where  $\sigma_{ij}$  and  $\varepsilon_{ij}$  are the second-order stress and strain tensors respectively,  $Q_{ijkl}$  is the fourth-order elastic stiffness tensor,  $d_{klm}$  is the third-order piezoelectric strain tensor and  $E_m$  is the applied electrical field.

For a structure whose length is large compared to its width and thickness the Euler–Bernoulli theory can be utilized for the solutions at all locations away from the edges. Further, if the voltage is applied across the thickness the electrical field is generated only in the thickness direction and is denoted by  $E_z$ . In such a case, Eq. (10) reduces to

$$\sigma_x = Q(\varepsilon_x - d_{31}E_z) \quad (11)$$

where  $Q$  is the elastic modulus and  $d_{31}$  is the piezoelectric strain coefficient. The strain at any point ( $\varepsilon_x$ ) can be expressed in terms of strain at the reference plane as follows:

$$\varepsilon_x = \varepsilon_o - z\kappa \quad (12)$$

where,  $\varepsilon_o$  is the in-plane strain and  $\kappa$  is the curvature.

Using the variable definitions from Fig. 1, the stress–strain relation in Eq. (11) for different layers can be elaborated as

$$\sigma_x = \begin{cases} (Q_s)[\varepsilon_o - z\kappa - d_{31}^s E_z] & -h_1 \leq z \leq h_1 \\ (Q_s + \Delta Q N(z))[\varepsilon_o - z\kappa \\ - (d_{31}^s + \Delta d_{31} N(z))E_z] & -h_1 \leq z \leq -h_2, h_1 \leq z \leq h_2 \\ (Q_a)[\varepsilon_o - z\kappa - d_{31}^a E_z] & -h_2 \leq z \leq -h_3, h_2 \leq z \leq h_3 \end{cases} \quad (13)$$

where  $Q_s$  and  $Q_a$  are the elastic moduli and  $d_{31}^s$  and  $d_{31}^a$  are the piezoelectric strain coefficients of the substrate and the active material respectively. The mismatches are given as

$$\Delta Q = (Q_a - Q_s)$$

$$\Delta d_{31} = (d_{31}^a - d_{31}^s)$$

The strain components  $\varepsilon_o$  and  $\kappa$  are obtained using the force and bending moment equilibrium equations,

$$\mathbf{K}\boldsymbol{\varepsilon} + \mathbf{B} = 0 \quad (14)$$

where  $\mathbf{K}$  is a  $2 \times 2$  matrix of rigidity coefficients,  $\boldsymbol{\varepsilon} = (\varepsilon_o - \kappa)^T$  is the vector of strains and,

$\mathbf{B} = (b_1 \ b_2)^T$  is the vector of body forces developed due to applied electrical field. The terms in  $\mathbf{K}$  and  $\mathbf{B}$  are defined as

$$k_{11} = \sum [Q_s h_1 + Q_s (h_2 - h_1) + \Delta Q I_1 + Q_a (h_3 - h_2)] \quad (15)$$

$$k_{12} = k_{21} = \sum \left[ \frac{Q_s h_1^2}{2} + \frac{Q_s (h_2^2 - h_1^2)}{2} + \Delta Q I_2 + \frac{Q_a (h_3^2 - h_2^2)}{2} \right] \quad (16)$$

$$k_{22} = \sum \left[ \frac{Q_s h_1^3}{2} + \frac{Q_s (h_2^3 - h_1^3)}{2} + \Delta Q I_3 + \frac{Q_s (h_3^3 - h_2^3)}{2} \right] \quad (17)$$

$$b_1 = \sum [Q_s E_z (d_{31}^s h_1 + d_{31}^s (h_2 - h_1) + \Delta d_{31} I_1) + \Delta Q E_z (d_{31}^s I_1 + \Delta d_{31} I_3) + (Q_a (h_3 - h_2) d_{31}^a E_z)] \quad (18)$$

$$b_2 = \sum \left[ \frac{Q_s E_z}{2} (d_{31}^s h_1^2 + 2\Delta d_{31} I_4) + \Delta Q E_z (d_{31}^s I_4 + \Delta d_{31} I_6) + \left( \frac{Q_a d_{31}^a E_z}{2} (h_3^2 - h_2^2) \right) \right] \quad (19)$$

The integral terms ( $I_1$ – $I_6$ ) in Eqs. (15)–(19) depend on the compositional gradation. For power-law gradation the terms are given as

$$I_1 = \int_{h_1}^{h_2} N(z) dz = \frac{1}{(h_2 - h_1)^n} \left[ \frac{(h_2 - h_1)^{n+1}}{n+1} \right] \quad (20a)$$

$$I_2 = I_4 = \int_{h_1}^{h_2} N(z) z dz = \frac{1}{(h_2 - h_1)^n} \left[ \frac{(h_2 - h_1)^{n+2}}{n+2} - \frac{h_1 (h_2 - h_1)^{n+1}}{n+1} \right] \quad (20b)$$

$$\begin{aligned}
 I_3 &= \int_{h_1}^{h_2} (N(z))^2 dz \\
 &= \frac{1}{(h_2 - h_1)^{2n}} \left[ \frac{(h_2 - h_1)^{2n+1}}{2n+1} \right] \quad (20c)
 \end{aligned}$$

$$\begin{aligned}
 I_5 &= \int_{h_1}^{h_2} N(z)z^2 dz \\
 &= \frac{1}{(h_2 - h_1)^n} \left[ \frac{(h_2 - h_1)^{n+3}}{n+3} \right. \\
 &\quad \left. + h_1 \frac{(h_2 - h_1)^{n+2}}{n+2} + h_1^2 \frac{(h_2 - h_1)^{n+1}}{n+1} \right] \quad (20d)
 \end{aligned}$$

$$\begin{aligned}
 I_6 &= \int_{h_1}^{h_2} (N(z))^2 z dz \\
 &= \frac{1}{(h_2 - h_1)^{2n}} \left[ \frac{(h_2 - h_1)^{2n+2}}{2n+2} + h_1 \frac{(h_2 - h_1)^{2n+1}}{2n+1} \right] \quad (20e)
 \end{aligned}$$

For the CG2 the integral terms are as follows:

$$I_1 = \frac{(h_2 - h_1)}{2} \quad (21a)$$

$$I_2 = I_4 = (h_2 - h_1) \left\{ \frac{7(h_2 - h_1)}{20} + \frac{h_1}{2} \right\} \quad (21b)$$

$$I_3 = \frac{13}{35}(h_2 - h_1) \quad (21c)$$

$$I_5 = (h_2 - h_1) \left\{ \frac{4(h_2 - h_1)^2}{15} + \frac{14(h_2 - h_1)h_1}{20} + \frac{h_1^2}{2} \right\} \quad (21d)$$

$$\begin{aligned}
 I_6 &= (h_2 - h_1) \left[ \left( \frac{9h_1}{5} + \frac{9(h_2 - h_1)}{6} \right) \right. \\
 &\quad \left. - \left( 2h_1 + \frac{12(h_2 - h_1)}{7} \right) + \left( \frac{4h_1}{7} + \frac{(h_2 - h_1)}{2} \right) \right] \quad (21e)
 \end{aligned}$$

In Eqs. (15)–(19) the  $\sum$  sign indicates summation of the terms over the thickness of the material. It may be noted that Eqs. (21a–e) and (22a–e) are derived for the positive  $z$ -coordinate. Similar ex-

pressions can be obtained for the negative  $z$ -coordinate. The terms  $\varepsilon_o$  and  $\kappa$  are obtained solving Eq. (14). Once the solution for  $\varepsilon_o$  and  $\kappa$  is available, the stress distribution over the thickness of material is obtained using Eq. (13).

### 3. Results and discussion

The influence of different electro-elastic grading on the stress distribution in the material is analyzed. For the sake of comparison, linear and cubic power-law grading are considered along with the cubic grading with controlled gradients (CG2). Keeping the piezoelectric mismatch constant, the modular ratios ( $Q_s/Q_a$ ) ranging between 0.05 and 10.0 are considered in order to cover a broad range of constituent phases. Unless otherwise specified, the substrate is assumed electrically inactive ( $d_{31}^s = 0$ ).

To characterize the material a balanced lay-up of the constituents has been considered. In a balanced composite the in-plane strain  $\varepsilon_o$  and the curvature  $\kappa$  Eq. (11) can be created separately. When an electrical field of the same magnitude and polarity is applied at both the active layers only in-plane strain exists. Only curvature is created through the application of equal electrical fields but with opposite polarities. In the framework of small deformation theory, a superposition of both cases gives response of the material subjected to an arbitrary electrical field. In all the cases, an electrical field inducing a strain ( $d_{31}^a E_z$ ) of 100 *microstrain* is applied. It is to be noted that the force applied will be different for different compositional gradients. The stress variation is plotted as a non-dimensional ratio of the axial stress to the stress induced in a pure piezoelectric material ( $Q_a d_{31}^a E_z$ ). This leads to a qualitative description of the stress variation that is independent of the actual material properties and induced strain.

#### 3.1. Example 1—in-plane strain

Figs. 4a–b and 5a–b show a comparison between the stress characteristics for linear ( $n = 1$ ) and CG2 FGM subjected to a pure membrane

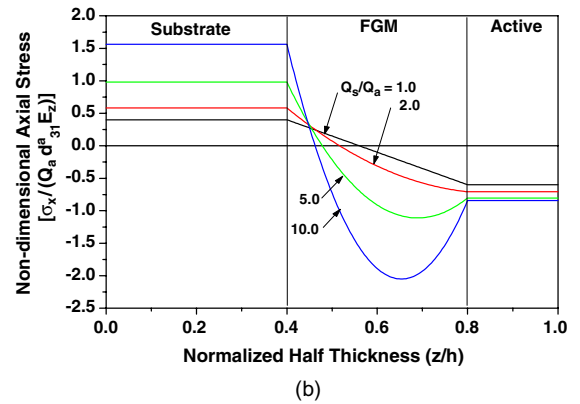
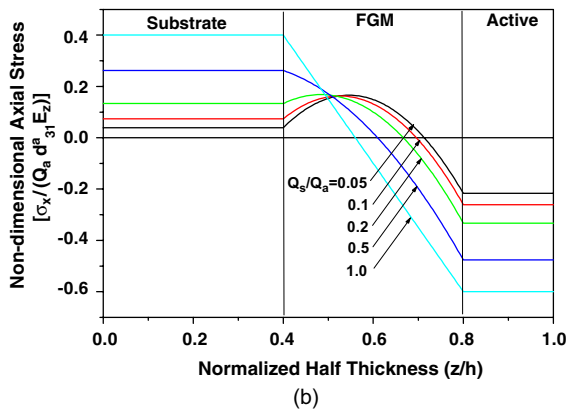
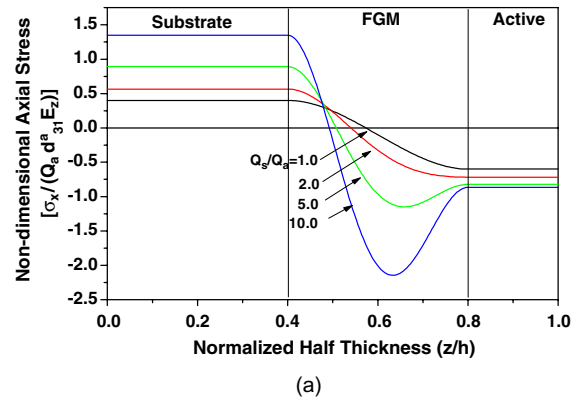
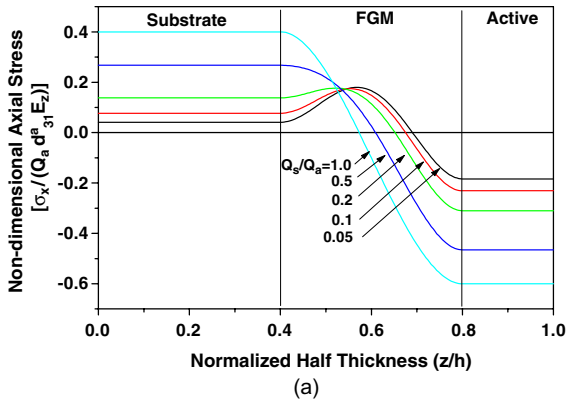


Fig. 4. Non-dimensional in-plane stress due to electric field of same polarity ( $Q_s/Q_a \leq 1.0$ ): (a) CG2, (b) linear gradation.

Fig. 5. Non-dimensional in-plane stress due to electric field of same polarity ( $Q_s/Q_a \geq 1.0$ ): (a) CG2, (b) linear gradation.

strain. The advantage of controlling the gradients of the gradation profile is clearly visible from the comparison of the stress profiles for linear and cubic variation. Sharp cusps at the interfaces of the FG layer are completely eliminated and the stress distribution is smoother.

Further, the CG2 results in stress reduction at the interface between the FG layer and stiffer layer. Maximum reduction in the stress occurs for extreme modular ratios and it is about 16% for modular ratio 0.05 (Fig. 6).

The CG2 not only results in reduced stresses but also minimizes the maximum stress gradient resulting in better overall stress distribution in the material. In Fig. 7, zero stress gradients at the interfaces in case of CG2 indicate smooth stress distribution. Linear grading results in discontinuity

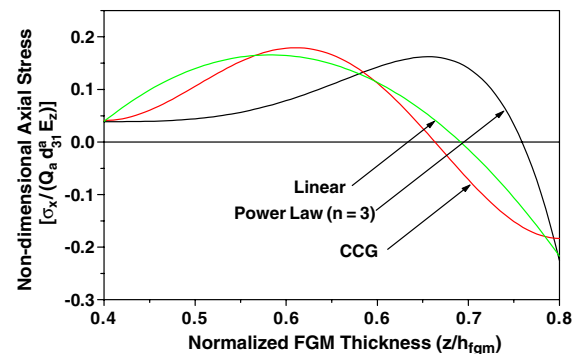


Fig. 6. Comparison of axial stresses for modular ratio 0.05.

in the slope of the stress distribution at both interfaces whereas cubic power-law grading possesses high stress gradient at the FG–active layer

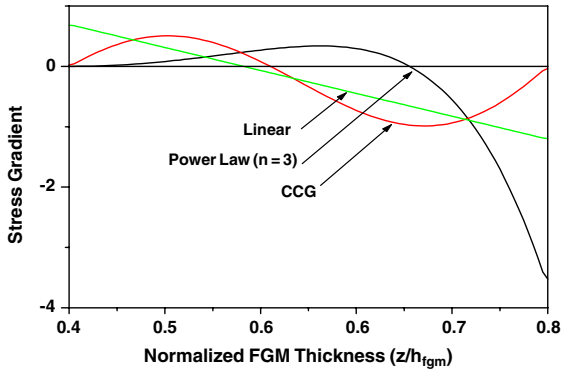


Fig. 7. Comparison of stress gradients in the FG layer for modular ratio 0.05.

interface. The CG2 leads to about 17% reduction in the stress gradient over linear FGM and nearly 72% lower as compared to the cubic power-law FGM. Albeit smooth, a high-order power-law performs poorly because the nature of grading results in sharp change in the material characteristics over small region in the FG layer, which is undesirable (Fig. 2). For CG2, the maximum stress gradient occurs at 0.75 times the thickness of FG layer, whereas the gradient is highest at the FG–stiff layer interface for a power-law FGM.

3.2. Example 2—curvature strain

Figs. 8a–b and 9a–b present the non-dimensional stress distribution due to the application of electrical fields with opposite polarities in the active layers. In this case a pure curvature is produced in the material characterized by linear variation of stresses in the non-FG layers. In the FG layer, however, the variation is nonlinear. All the observations made in case of the previous example are also valid here.

4. Conclusions

This paper discusses the stress characteristics of an active FGM subjected to electrical field. An exact solution based on Euler–Bernoulli theory that includes electro-mechanical constitutive rela-

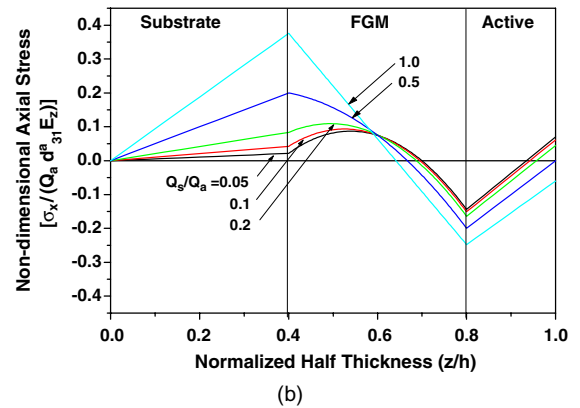
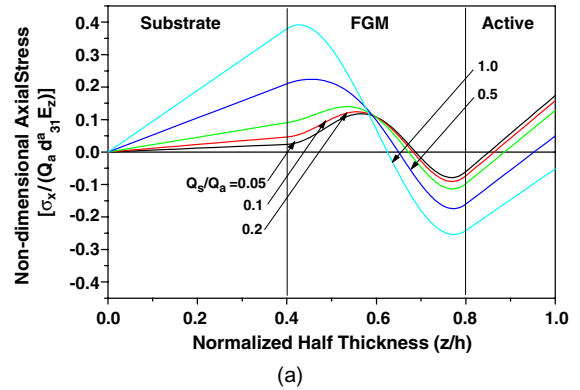


Fig. 8. Non-dimensional in-plane stress due to electric field of opposite polarity ( $Q_s/Q_a \leq 1.0$ ): (a) CG2, (b) linear gradation.

tionship is presented. Non-dimensional graphs are presented for the qualitative description of stresses over material thickness. Large difference in the elastic properties of the constituent material phases results in stress distribution with steep gradients near the stiffer material. As the nature of the stress depends on the polarity of the applied electrical field, high tensile stress may be induced near the stiffer material. This situation needs critical evaluation especially under dynamic conditions where repeated alternating of stresses may lead to fatigue and subsequently fracture of the material. The power-law based compositional gradations, which are incomplete polynomials, result in high stress-gradients at the interfaces of the FG layer. The interfacial stress gradients are completely avoided by employing CG2. Owing to  $C^1$ -

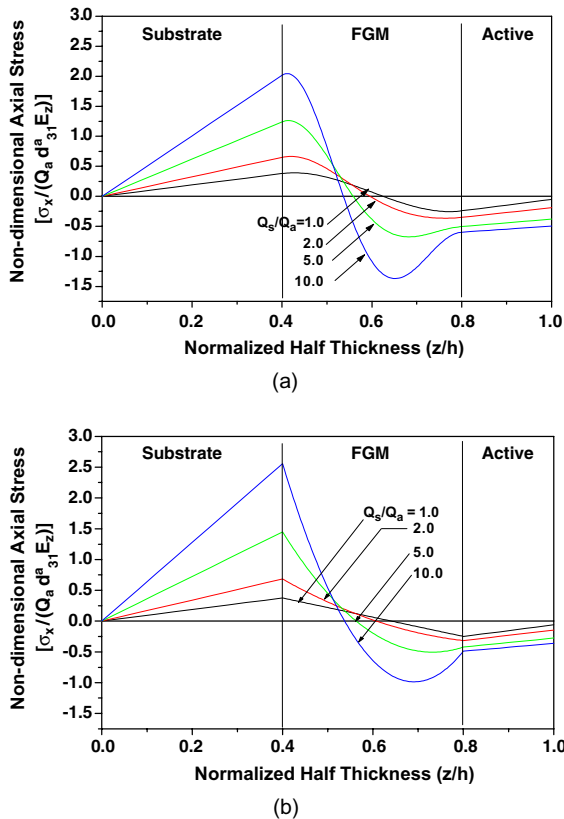


Fig. 9. Non-dimensional in-plane stress due to electric field of opposite polarity ( $Q_s/Q_a \geq 1.0$ ): (a) CG2, (b) linear gradation.

continuity, the CG2 not only eliminates stress gradients at the interfaces but also reduces the maximum gradient occurring over the thickness of the material. Future work includes examining the energy transduction characteristics for different compositional variations and investigations on the two-dimensional cases with mismatch in the electrical and the elastic properties.

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